Tutorial Sheet-3: Proof Techniques

(1) Prove that if n is an integer and n^2 is odd, then n is odd.

(2) Prove that $\sqrt{p}$ is irrational, where p is prime number.

(3) Show that the square of an even number is an even number.

(4) Prove or disprove that the product of two irrational numbers is irrational.

(5) Prove that if x is rational and $x \neq 0$, then $\frac{1}{x}$ is rational.

(6) Prove that sum of a rational number and an irrational number is irrational.

(9) Prove or disprove $7^n - 4^n$ is divisible by 3, for all $n \in \mathbb{N}$.

(10) Prove or disprove $9(9^n - 1) - 8n$ is divisible by 64, for all $n \in \mathbb{N}$.

(11) Prove that $\arctan \frac{1}{3} + \arctan \frac{1}{7} + \cdots + \arctan \frac{1}{n^2+n+1} = \arctan \frac{n}{n+2}$, for all $n \in \mathbb{N}$.

(12) Prove that if n is an integer, then $n^2 \geq n$.

(13) Show that if a and b are integers and both ab and $a + b$ are even, then both a and b are even.

(14) Prove that $m^2 = n^2$ if and only if $m = n$ or $m = n$.

(15) Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

(16) Prove that if n is a perfect square, then $n + 2$ is not a perfect square.

(18) Let $n \in \mathbb{N}$ and suppose we are given real numbers $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. Then Arithmetic mean (AM) $= \frac{a_1+a_2+\ldots a_n}{2} \geq (a_1 a_2 \ldots a_n)^{\frac{1}{n}} =$ GM (Geometric mean).

(19) Fix a positive integer n and let A be a set with $|A| = n$. Let $P(A)$ denote the power set of A. Then show that $|P(A)| = 2^n$.

(20) Consider the following recursively defined set S :

1. $a \in S$
2. If $x \in S$, then $(x) \in S$

Prove by structural induction that every element in S contains an equal number of right and left parentheses.

(21) Prove the following property about the length function: $\forall\ y, x \in X, \quad len(xy) = len(x) + len(y)$, where X is the collection of all finite strings on the finite alphabet set.

(22) Consider the function $f : \{\ldots, -3, -2, -1\} \rightarrow \{\ldots, -3, -2, -1\}$ defined recursively as follows: $f(-1) = -1$, $f(n) = f(n + 1) + n$ for $n < -1$. Show that $f(n) = -\frac{|n|(|n|+1)}{2}$.